CH 10 – MAXIMIZING COMPANY PROFIT

❑ INTRODUCTION

The only good profit is the best profit. Selling just a few widgets clearly does not maximize our profit, but selling too many widgets may also not result in our best profit, due to increased costs. In other words, sometimes we seek an optimal number of widgets to sell. Selling either more or less than this number will result in reduced profit.

❑ ALGEBRA REVIEW

Recall that "$x < 8$" means that x is <u>less than</u> 8, and the phrase "$y > 17$" means that y is <u>greater than</u> 17.

If we want to say that x is greater than 9 but less than 15, we write

$$9 < x < 15$$

This can also be read "x is between 9 and 15, but not equal to either of them." If we assume that x is a whole number, then $9 < x < 15$ means that x must be one of the numbers 10, 11, 12, 13, or 14. But if there's no restriction on x (that is, if x can be any real number), then there are infinitely many choices for x, including rational numbers like 9.001, $10\frac{1}{4}$, 13.83, $14\frac{101}{102}$, and even irrational numbers like 3π and $\sqrt{101}$.

Homework

1. Consider the double inequality $12 < a < 20$, where a is a real number. Which of the following numbers are possible values of a?

11.9	12	12.001	19.9	20	4π
$\sqrt{410}$	$\sqrt{300}$	$\sqrt{140}$	7π	$10\frac{7}{8}$	0

2. Use inequality signs to rewrite each statement:

   a. n is less than 7
   b. z is larger than 0
   c. a is smaller than 12
   d. c is greater than -4
   e. x is between 5 and 7, and is equal to neither of them
   f. y is between -3 and 3, and is equal to neither of them

3. a. $13^2 =$ b. $(-9)^2 =$ c. $-3^2 =$ d. $-10^2 =$
   e. $(-12)^2 =$ f. $-12^2 =$ g. $(-7)^2 =$ h. $-7^2 =$

4. a. $(-5)^2 - 2(3) + 1 =$
   b. $-5^2 + 2(-3) - 1 =$
   c. $-6^2 - 5(-6) + 3 =$
   d. $-(-4)^2 - 3(-1) + 10 =$
   e. $-3^2 - 2(3) - 3 =$
   f. $-(-1)^2 + 4(7) + 1 =$
   g. $-10^2 - 4(-3) - (-6) =$
   h. $-(-2)^2 - 3(4) + 12 =$
   i. $(-7)^2 - 7(-7) + 2(-1) =$
   j. $4^2 -(9)(-1) + (-3) =$

5. Evaluate the given expression for the given value of x:

   a. $x^2 + 7x + 1 \quad x = 5$
   b. $2x^2 - x - 3 \quad x = -3$
   c. $3x^2 - 8x \quad x = 0$
   d. $9x^2 + 7 \quad x = 0$
   e. $-2x^2 + 3x - 4 \quad x = -3$
   f. $-3x^2 - 2x - 1 \quad x = -2$

g. $-x^2 + 2x - 1$ $\quad x = 4$ $\qquad$ h. $-x^2 - 4x + 5$ $\quad x = -10$

i. $-4x^2 - x + 3$ $\quad x = -5$ $\qquad$ j. $-x^2 + 7x - 12$ $\quad x = 13$

❑ PROFIT AND BREAK-EVEN POINTS REVISITED

EXAMPLE 1: **If the revenue is given by the formula $R = 2w^2 - 6w + 9$ and the cost formula is given by $C = w^2 + 7w - 10$, find the profit formula in simplest form.**

Solution: We have our basic formula for profit which says that profit is the difference between revenue and cost:

$$P = R - C \qquad \text{(the profit formula)}$$

$$\Rightarrow \quad P = (2w^2 - 6w + 9) - (w^2 + 7w - 10) \qquad \text{(given formulas)}$$

$$\Rightarrow \quad P = 1(2w^2 - 6w + 9) - 1(w^2 + 7w - 10) \qquad \text{(this step is } \textit{optional}\text{)}$$

$$\Rightarrow \quad P = 2w^2 - 6w + 9 - w^2 - 7w + 10 \qquad \text{(distribute)}$$

$$\Rightarrow \quad \boxed{P = w^2 - 13w + 19} \qquad \text{(combine like terms)}$$

Homework

6. For the given revenue and cost formulas, calculate the **profit** formula in simplest form:

a. $R = 3w^2 - 6w + 1$ $\qquad C = 2w^2 + 6w - 31$

b. $R = 5w^2 + 8w - 10$ $\qquad C = 4w^2 + 8w + 90$

c. $R = 8w^2 - 55$ $\qquad C = 6w^2 + 22w - 1$

d. $R = 2w^2 + 3w + 3$ $\qquad C = w^2 + 2w$

EXAMPLE 2: **Consider the profit formula $P = w^2 - 9$. Graph this formula and determine the break-even point.**

Solution: Let's calculate the values of the profit formula for a few values of w. We'll include some negative values of w so we get a better picture of the graph, but note that a break-even point can never be negative, since we can't actually produce a negative number of widgets.

$w = 0 \Rightarrow P = \mathbf{0}^2 - 9 = -9$ (a loss)

$w = 1 \Rightarrow P = \mathbf{1}^2 - 9 = 1 - 9 = -8$ (a loss)

$w = 3 \Rightarrow P = \mathbf{3}^2 - 9 = 9 - 9 = 0$ (a *break-even point*)

$w = 4 \Rightarrow P = \mathbf{4}^2 - 9 = 16 - 9 = 7$ (a real profit!)

$w = -2 \Rightarrow P = (\mathbf{-2})^2 - 9 = 4 - 9 = -5$

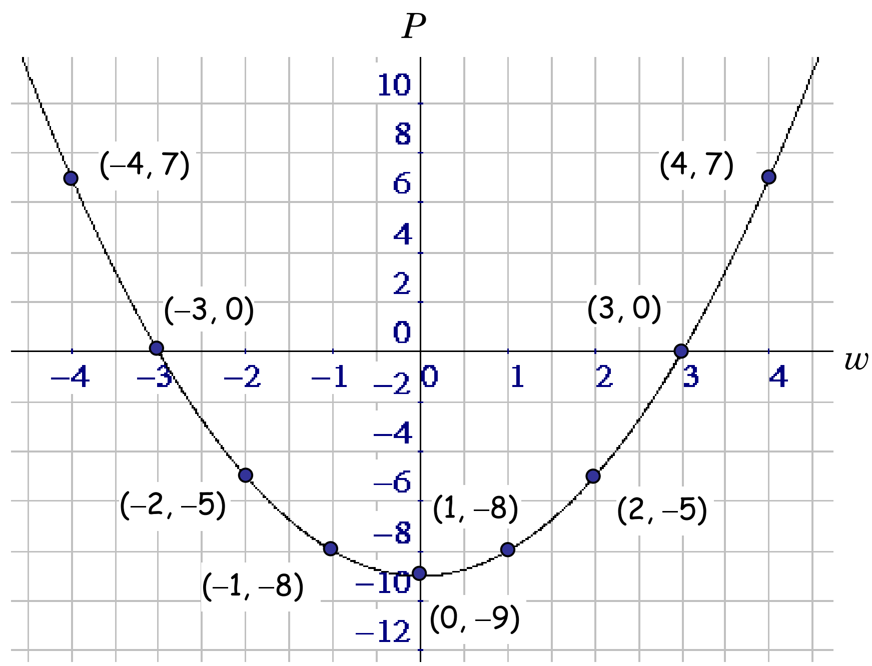
$w = -4 \Rightarrow P = (\mathbf{-4})^2 - 9 = 16 - 9 = 7$

These calculations, along with a few others for you to confirm, give us the following table:

w	−4	−3	−2	−1	0	1	2	**3**	4
P	7	0	−5	−8	−9	−8	−5	**0**	7

The break-even point is 3 widgets, which produces a profit of $0.

We can make some deductions now. (Remember, we're looking at positive values of w only.) If we sell 2 or fewer widgets, we incur a loss. At 3 widgets our profit is 0, and thus $w = 3$ is the break-even point. Four widgets and up, we make a profit. Now it's time for our graph. By plotting some points in the table above, we get our picture:



$$P = w^2 - 9$$

Notice that this graph is not a line. The exponent of 2 in the formula makes it a curvy graph called a ***parabola*** (puh-RAB-oh-luh).

What can we tell from the right-half of the graph (Quadrants I and IV)? When $w < 3$, the graph is below the w-axis, so we're running a loss; when $w = 3$, the profit is zero (on the w-axis), and so this is the break-even point; and when $w > 3$, the graph is above the w-axis, implying a positive profit, and we're finally making money. It also appears from our graph that as w grows larger, so does our profit.

<u>EXAMPLE 3:</u> **Graph the parabola $y = x^2 - 4$.**

a) Where does the graph cross the x-axis?

b) Where does the graph cross the y-axis?

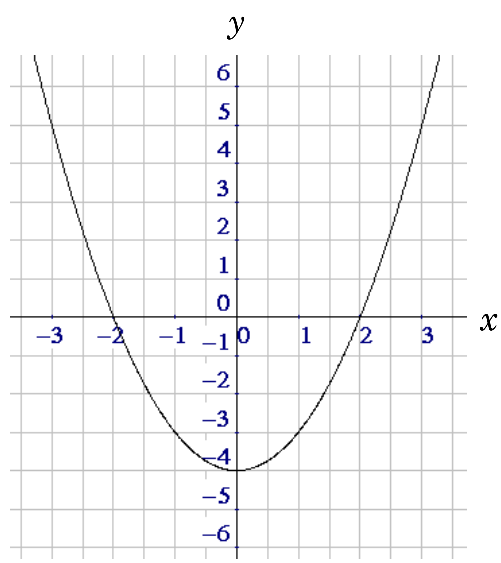
c) Find the lowest point on the parabola.

Solution: Here are a few calculations:

$$x = -3 \;\Rightarrow\; y = (\mathbf{-3})^2 - 4 = 9 - 4 = \underline{5} \;\Rightarrow\; (-3, 5)$$
$$x = -2 \;\Rightarrow\; y = (\mathbf{-2})^2 - 4 = 4 - 4 = \underline{0} \;\Rightarrow\; (-2, 0)$$
$$x = 0 \;\Rightarrow\; y = \mathbf{0}^2 - 4 = 0 - 4 = \underline{-4} \;\Rightarrow\; (0, -4)$$
$$x = 3 \;\Rightarrow\; y = \mathbf{3}^2 - 4 = 9 - 4 = \underline{5} \;\Rightarrow\; (3, 5)$$

You should now check all the values in the following table:

x	–3	–2	–1	0	1	2	3
y	5	0	–3	–4	–3	0	5



a) The graph crosses the x-axis at the points (2, 0) and (–2, 0).

b) The graph crosses the y-axis at the point (0, –4).

c) The lowest point on the graph is the point (0, –4).

<u>EXAMPLE 4:</u> **Graph the parabola $z = -u^2 - 4u + 5$.**

a) Where does the graph cross the u-axis?

b) Where does the graph cross the z-axis?

c) Find the highest point on the parabola.

<u>Solution:</u> There's no way to tell for sure which variable, u or z, goes on which axis. But the given formula indicates that the value of z depends on the value of u, so our best bet is that u should go on the horizontal axis and z on the vertical axis. Here are some calculations. As before, verify these calculations and those in the table which follows:

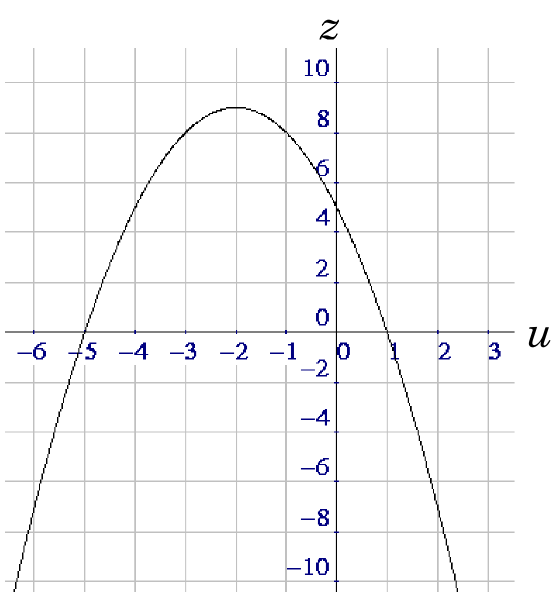
$$u = -5 \;\Rightarrow\; z = -(\mathbf{-5})^2 - 4(\mathbf{-5}) + 5 = -25 + 20 + 5 = \underline{0}$$

which gives the point $(-5, 0)$.

$$u = 2 \;\Rightarrow\; z = -(\mathbf{2})^2 - 4(\mathbf{2}) + 5 = -4 - 8 + 5 = \underline{-7}$$

which gives the point $(2, -7)$.

u	-6	-5	-4	-3	-2	-1	0	1	2
z	-7	0	5	8	9	8	5	0	-7



a) The graph crosses the u-axis at the points $(-5, 0)$ and $(1, 0)$.

b) The graph crosses the z-axis at the point $(0, 5)$.

c) The highest point on the graph is the point $(-2, 9)$.

EXAMPLE 5: **Consider the profit formula**

$$P = -w^2 + 10w - 21$$

Determine the two break-even points and also determine the selling level (the value of *w*) which will produce the maximum (highest) profit.

Solution: We start with a table and try to answer the questions. Then in the homework you'll graph the formula and try to answer the questions again.

$$w = 1 \Rightarrow P = -(\mathbf{1})^2 + 10(\mathbf{1}) - 21 = -1 + 10 - 21 = \underline{-12}$$

$$w = 3 \Rightarrow P = -(\mathbf{3})^2 + 10(\mathbf{3}) - 21 = -9 + 30 - 21 = \underline{0}$$

$$w = 6 \Rightarrow P = -(\mathbf{6})^2 + 10(\mathbf{6}) - 21 = -36 + 60 - 21 = \underline{3}$$

You do the calculations to fill in the rest of the table:

w	1	2	3	4	5	6	7	8
P	–12	–5	0	3	4	3	0	–5

Since a break-even point occurs when the profit is zero, we see that there are actually two break-even points, $w = 3$ and $w = 7$. We can also see that the table tells us we have a loss when $w < 3$, we have a profit when $3 < w < 7$, and we incur a loss if $w > 7$.

By the way, how can selling more than 7 widgets cause a loss in the business? Perhaps that much selling requires an extra shift at the factory to make the widgets, or perhaps we need a new warehouse to store all the widgets. Whatever the reason, many businesses have gone under trying to pursue that "one extra dollar."

Now for the second part of the question. Finding the value of w which produces the maximum profit means we should look for the highest number in the second row of the table, the profit row. Clearly, the maximum profit is $P = 4$, and this occurs when $w = 5$.

Thus, we say that

> The two break-even points are $w = 3$ and $w = 7$. And, a selling level of 5 widgets produces the maximum profit of $4.

Homework

7. For each profit formula, use a table to find the **break-even point**.

   a. $P = w^2 - 16$ b. $P = -w^2 + 4$

   c. $P = 25 - w^2$ d. $P = w^2 - 8w + 12$

   e. $P = -w^2 + 6w - 5$ f. $P = 2w^2 - 5w - 3$

8. Graph the profit formula in Example 5, and verify the final results of the example. By the way, the very highest (or lowest) point on a parabola is called its ***vertex***.

9. For each profit formula, use a table — but verify with a graph — to find the value of *w* which will produce the **maximum profit**:

   a. $P = -w^2 + 6w - 5$ b. $P = -w^2 + 10w$

   c. $P = -w^2 + 8w + 7$ d. $P = -2w^2 + 8w + 1$

10. Graph each parabola:

a. $y = x^2 + 2x$

b. $y = x^2 - 4x$

c. $y = x^2 - 1$

d. $y = -x^2 + 4$

e. $y = x^2 + 6x + 9$

f. $y = x^2 - 4x + 4$

g. $y = -x^2 + 10x - 25$

h. $y = -x^2 + 6x - 5$

Practice Problems

11. If the revenue is given by the formula $R = 2w^2 - 6w + 9$ and the cost formula is given by $C = w^2 + 7w - 10$, find the profit formula in simplest form.

12. Consider the profit formula $P = -w^2 + 8w - 12$.

    a. Graph the profit formula.
    b. Find the two break-even points.
    c. Find the number of widgets which will produce the maximum profit.
    d. What is the maximum profit?

13. Graph the parabola $y = x^2 + 4x + 4$. Find all the intercepts and the vertex.

14. If $P = w^2 - 12w + 20$ is the profit formula, find the break-even points.

15. Graph the parabola $y = 0.2x^2 - 7$. Use the graph to estimate the x-intercepts.

Solutions

1. 12.001 19.9 4π $\sqrt{300}$

2. a. $n < 7$ b. $z > 0$ c. $a < 12$
d. $c > -4$ e. $5 < x < 7$ f. $-3 < y < 3$

3. a. 169 b. 81 c. –9 d. –100
e. 144 f. –144 g. 49 h. –49

4. a. 20 b. –32 c. –3 d. –3 e. –18
f. 28 g. –82 h. –4 i. 96 j. 22

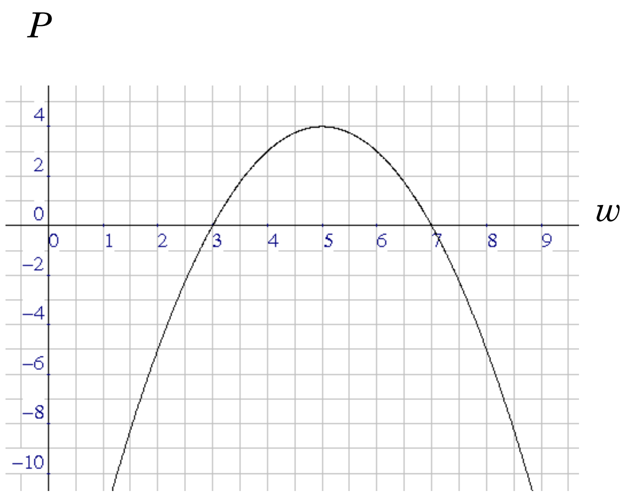
5. a. 61 b. 18 c. 0 d. 7 e. –31
f. –9 g. –9 h. –55 i. –92 j. –90

6. a. $P = R - E = (3w^2 - 6w + 1) - (2w^2 + 6w - 31)$
$= 3w^2 - 6w + 1 - 2w^2 - 6w + 31 = w^2 - 12w + 32$

b. $P = w^2 - 100$ c. $P = 2w^2 - 22w - 54$ d. $P = w^2 + w + 3$

7. a. $w = 4$ b. $w = 2$ c. $w = 5$
d. $w = 2$ and $w = 6$ e. $w = 1$ and $w = 5$ f. $w = 3$

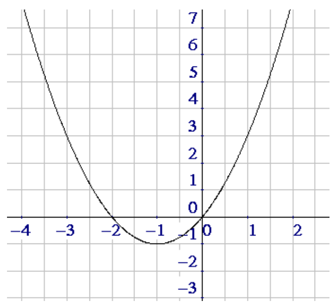
8.



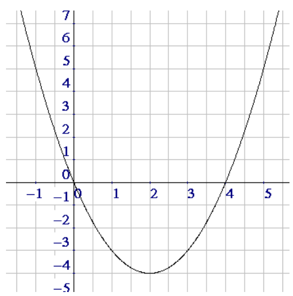
The break-even points are the w values where the profit is zero. This occurs on the w-axis, where the graph crosses at (3, 0) and (7, 0). Thus, the break-even points are $w = 3$ and $w = 7$. As for $w = 5$ being the optimal number of widgets to sell, this is indicated by noticing that the vertex of the parabola is at (5, 4). This means that 5 widgets will produce the most profit, namely $4.

9. a. $w = 3$ b. $w = 5$ c. $w = 4$ d. $w = 2$

10. a.



b.



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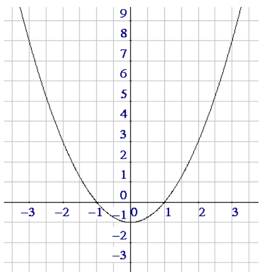

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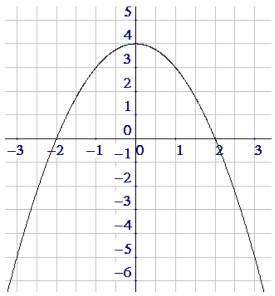

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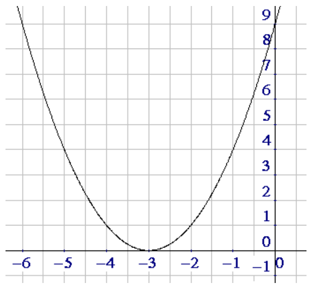

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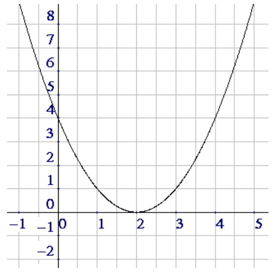
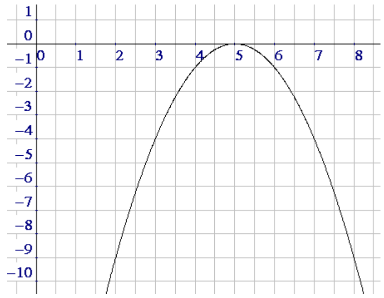


g.



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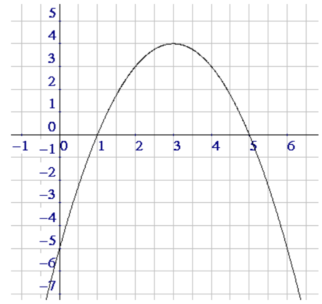
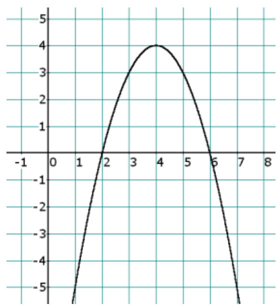


11. $P = w^2 - 13w + 19$

12. a.



b. $w = 2$, $w = 6$

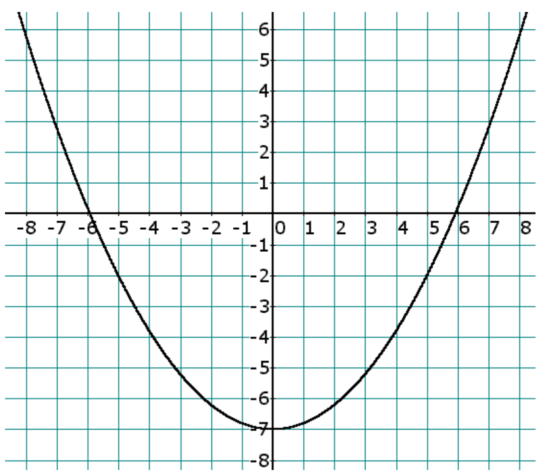
c. $w = 4$

d. $4

13. x: (–2, 0) y: (0, 4) V(–2, 0)

14. $w = 2$, $w = 10$

15.



The x-intercepts are approximately (5.9, 0) and (–5.9, 0)

"It is in fact a part of the function of education to help us escape, not from our own time – for we are bound by that – but from the intellectual and emotional limitations of our time."

– T. S. Eliot